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On the symmetric subscheme of Hilbert scheme of points

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Abstract

We consider the Hilbert scheme $\mathrm{Hilb}^{d+1}(\mathbb{C}^d)$ of $(d+1)$ points in affine d-space $\mathbb{C}^d$ $(d \geq 3)$, which includes the square of any maximal ideal. We describe equations for the most symmetric affine open subscheme of $\mathrm{Hilb}^{d+1}(\mathbb{C}^d)$, in terms of Schur modules. In addition we prove that $\mathrm{Hilb}^n(\mathbb{C}^d)$ is reducible for $n > d \geq 12$.

Keywords : Hilbert schemes, Ideal projectors, Littlewood-Richardson rule

1 Introduction

Throughout these notes we work over $\mathbb{C}$. The maximal ideals in a polynomial ring are very basic objects, and their deformations are easy to understand. However very little is known about the family of the ideals that can be deformed to the square of a maximal ideal. Its existence and connectedness [8] are well known. Here we study its dimension.

We consider the Hilbert scheme $\mathrm{Hilb}^{d+1}(\mathbb{C}^d)$ of $(d+1)$ points in affine d-space $\mathbb{C}^d$, $d \geq 3$ (for general introduction to the Hilbert schemes of points, see [11, §18.4]). It parametrizes the ideals I of colength $(d+1)$ in $\mathbb{C}[\mathbf{x}] = \mathbb{C}[x_1, ..., x_d]$. As with any moduli problem, it is natural to ask whether $\mathrm{Hilb}^{d+1}(\mathbb{C}^d)$ is irreducible. It is already interesting because $\mathrm{Hilb}^{d+1}(\mathbb{C}^d)$ is irreducible for $d \leq 3$ but reducible for $d \geq 12$.[1]

Theorem A. $\mathrm{Hilb}^n(\mathbb{C}^d)$ *is reducible for* $n > d \geq 12$.

Our main purpose is to describe equations for the most symmetric affine open subscheme of $\mathrm{Hilb}^{d+1}(\mathbb{C}^d)$. We let $U \subset \mathrm{Hilb}^{d+1}(\mathbb{C}^d)$ denote the affine open subscheme consisting of all ideals $I \in \mathrm{Hilb}^{d+1}(\mathbb{C}^d)$ such that $\{1, x_1, ..., x_d\}$ is a $\mathbb{C}$-basis of $\mathbb{C}[\mathbf{x}]/I$. We will call U *the symmetric affine subscheme*. We note that the square of any maximal ideal in $\mathbb{C}[\mathbf{x}]$ belongs to the symmetric affine subscheme.

In these notes we give an elementary description of the coordinate ring of the symmetric affine subscheme U. For a $\mathbb{C}$-vector space V and a partition λ, the module $\mathbb{S}_\lambda V$ is defined by the Schur-Weyl construction. By abuse of notation, the quotient ring given by the ideal generated by $\mathbb{S}_\lambda V$ in the ring $\mathrm{Sym}^\bullet(\mathbb{S}_\mu V)$ for some partitions λ and μ will be denoted by $\frac{\mathrm{Sym}^\bullet(\mathbb{S}_\mu V)}{<\mathbb{S}_\lambda V>}$.

[1] Iarrobino [9] showed that $\mathrm{Hilb}^n(\mathbb{C}^d)$ is reducible for $d > 5$ and $n > (1+d)(1+d/4)$.

Theorem B. *Let $d \geq 3$. Let U be the symmetric affine open subscheme of* $\mathrm{Hilb}^{d+1}(\mathbb{C}^d)$. *Then U is isomorphic to*

$$\mathbb{C}^d \times \mathrm{Spec}\frac{\mathrm{Sym}^\bullet(\mathbb{S}_{(3,1,1,\cdots,1,0)}V)}{< \mathbb{S}_{(4,3,2,\cdots,2,1)}V >},$$

where V is a d-dimensional $\mathbb{C}$-vector space, $(3,1,1,\cdots,1,0)$ is a partition of $(d+1)$ and $(4,3,2,\cdots,2,1)$ is of $(2d+2)$.

Let us explain the notation more precisely. By Lemma 8, there is an injective homomorphism

$$j : \mathbb{S}_{(4,3,2,\cdots,2,1)}V \hookrightarrow \mathrm{Sym}^2(\mathbb{S}_{(3,1,1,\cdots,1,0)}V)$$

of Schur modules. Then j induces natural maps

$$\begin{aligned}
&\mathbb{S}_{(4,3,2,\cdots,2,1)}V \otimes \mathrm{Sym}^{r-2}(\mathbb{S}_{(3,1,1,\cdots,1,0)}V) \\
&\hookrightarrow \mathrm{Sym}^2(\mathbb{S}_{(3,1,1,\cdots,1,0)}V) \otimes \mathrm{Sym}^{r-2}(\mathbb{S}_{(3,1,1,\cdots,1,0)}V) \\
&\to \mathrm{Sym}^r(\mathbb{S}_{(3,1,1,\cdots,1,0)}V), \qquad\qquad r \geq 2,
\end{aligned}$$

which define the quotient ring $\frac{\mathrm{Sym}^\bullet(\mathbb{S}_{(3,1,1,\cdots,1,0)}V)}{\mathbb{S}_{(4,3,2,\cdots,2,1)}V}$.

Corollary C. *Let $H(r)$ be the Hilbert function of $\frac{\mathrm{Sym}^\bullet(\mathbb{S}_{(3,1,1,\cdots,1,0)}V)}{\mathbb{S}_{(4,3,2,\cdots,2,1)}V}$. Let*

$$\mathrm{Sym}^r(\mathbb{S}_{(3,1,1,\cdots,1,0)}V) = \bigoplus_{|\lambda|=r(d+1)} \mathbb{S}_\lambda^{\oplus m_\lambda},$$

where $m_\lambda \in \mathbb{Z}_{\geq 0}$ and $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_d)$ is a partition of $r(d+1)$, i.e., $\sum_{i=1}^d \lambda_i = r(d+1)$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d$. Then

$$H(r) \geq \sum_{\substack{|\lambda|=r(d+1) \\ \lambda_{d-k}+\cdots+\lambda_d \leq rk}} m_\lambda(\dim_{\mathbb{C}}\mathbb{S}_\lambda), \tag{1}$$

for any $r \geq 2$ and any $k = 0, ..., d-1$.

Corollary C is an elementary consequence of the combinatorial Littlewood-Richardson rule(for example, see [5, Appendix]). In fact any $\mathbb{S}_\lambda$ appearing in the decomposition of $\mathbb{S}_{(4,3,2,\cdots,2,1)}V \otimes (\mathbb{S}_{(3,1,1,\cdots,1,0)}V)^{\otimes(r-2)}$ satisfies $\lambda_{d-k} + \cdots + \lambda_d \geq rk+1$, for any $r \geq 2$ and any $k = 0, ..., d-1$.

It is tedious but entirely possible to compute the right hand side of (1) for small r. These computations suggest that the Hilbert function $H(r)$ grows faster than $\mathcal{O}\left(r^{k\binom{d-k}{2}}\right)$ for any $k = 0, ..., d-1$. So Corollary C suggests that, for sufficiently large d, the symmetric open subscheme U of $\mathrm{Hilb}^{d+1}(\mathbb{C}^d)$ has dimension greater than $d(d+1)$, which implies that $\mathrm{Hilb}^{d+1}(\mathbb{C}^d)$ is reducible. To prove Theorem A, we actually find large dimensional families of ideals in a very explicit way.

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2 Proof of Theorem B

To ease notations and references, we introduce the notion of ideal projectors(cf. [1], [3], [4], [14]).

Definition 1. (cf. [1]) A linear idempotent map P on $\mathbb{C}[\mathbf{x}]$ is called an **ideal projector** if $\ker P$ is an ideal in $\mathbb{C}[\mathbf{x}]$.

We will use *de Boor's formula*:

Theorem 2. ([3], de Boor) *A linear mapping $P : \mathbb{C}[\mathbf{x}] \to \mathbb{C}[\mathbf{x}]$ is an ideal projector if and only if the equality*

$$P(gh) = P(gP(h)) \tag{2}$$

holds for all $g, h \in \mathbb{C}[\mathbf{x}]$.

Let $\mathcal{P}$ be the space of ideal projectors onto span $\{1, x_1, ..., x_d\}$, in other words,

$$\mathcal{P} := \{P : \text{ideal projector} \mid \ker P \in U\}.$$

The space $\mathcal{P}$ is isomorphic to the symmetric affine subscheme U[12, p3]. For the sake of simplicity, we prefer to work on $\mathcal{P}$ in place of U.

First we consider the natural embedding of $\mathcal{P}$. Gustavsen, Laksov and Skjelnes [6] gave more general description of open affine coverings of Hilbert schemes of points.

Lemma 3. *The space $\mathcal{P}$ can be embedded into $\mathbb{C}^{(d+1)\binom{d+1}{2}}$.*

Proof. For each ideal projector $P \in \mathcal{P}$ and each pair (i, j), $1 \leq i, j \leq d$, there is a collection $p_{0,ij}, p_{1,ij}, \cdots, p_{d,ij}$ of complex numbers such that

$$P(x_i x_j) = p_{0,ij} + \sum_{m=1}^{d} p_{m,ij} x_m.$$

As (i, j) varies over $1 \leq i, j \leq d$, each ideal projector $P \in \mathcal{P}$ gives rise to a collection $p_{0,ij}, p_{k,st}$ $(1 \leq i, j, k, s, t \leq d)$ of complex numbers. Of course $p_{0,ij} = p_{0,ji}$ and $p_{k,st} = p_{k,ts}$. So we have a map $f : \mathcal{P} \to \mathbb{C}^{(d+1)\binom{d+1}{2}} = \frac{\mathbb{C}[p_{0,ij},\ p_{k,st}]_{1 \leq i,j,k,s,t \leq d}}{(p_{0,ij} - p_{0,ji},\ p_{k,st} - p_{k,ts})}$.

Here we only show that f is one-to-one. It is proved in [6] that f is in fact a scheme-theoretic embedding.

We will show that if $P_1, P_2 \in \mathcal{P}$ and if $f(P_1) = f(P_2)$, i.e. $P_1(x_i x_j) = P_2(x_i x_j)$ for every (i,j), $1 \le i,j \le d$, then $P_1 = P_2$. Since P_1 and P_2 are linear maps, it is enough to check that $P_1(x_{i_1}...x_{i_r}) = P_2(x_{i_1}...x_{i_r})$ for any monomial $x_{i_1}...x_{i_r}$. This follows from de Boor's formula (2):

$$\begin{aligned} P_1(x_{i_1}...x_{i_r}) &= P_1(x_{i_1} P_1(x_{i_2} \cdots P_1(x_{i_{r-1}} x_{i_r}) \cdots)) \\ &= P_2(x_{i_1} P_2(x_{i_2} \cdots P_2(x_{i_{r-1}} x_{i_r}) \cdots)) = P_2(x_{i_1}...x_{i_r}), \end{aligned}$$

where we have used the property that $P(g)$ is a linear combination of $1, x_1, \ldots, x_d$ for any $g \in \mathbb{C}[\mathbf{x}]$. $\square$

Next we describe the ideal defining $\mathcal{P}$ in

$$\frac{\mathbb{C}[p_{0,ij}, \quad p_{k,st}]_{1 \le i,j,k,s,t \le d}}{(p_{0,ij} - p_{0,ji}, \quad p_{k,st} - p_{k,ts})} =: R,$$

where we keep the notations in the above proof. Let $I_{\mathcal{P}}$ denote the ideal.

Lemma 4. *Let $C(a; j, (i,k)) \in R$ denote the coefficient of x_a in*

$$P(x_k P(x_i x_j)) - P(x_i P(x_k x_j)) \in R[x_1, \cdots, x_d].$$

Then $I_{\mathcal{P}}$ is generated by $C(a; j, (i,k))$'s ($0 \le a \le d, \ \ 1 \le i,j,k \le d$). (We regard an element in $R[x_1, \cdots, x_d]_0 \cong R$ as a coefficient of x_0.)

For example, if $a \ne j, i, k$ then

$$C(a; j, (i,k)) = \sum_{m=1}^{d} (p_{m,ij} p_{a,km} - p_{m,kj} p_{a,im}).$$

If $a = k$ then

$$C(k; j, (i,k)) = p_{0,ij} + \sum_{m=1}^{d} (p_{m,ij} p_{k,km} - p_{m,kj} p_{k,im}).$$

Proof of Lemma 4. The de Boor's formula (2) implies that $I_{\mathcal{P}}$ is generated by coefficients of x_a's ($1 \le a \le d$) in $P(gP(h)) - P(hP(g))$ (all $g, h \in \mathbb{C}[\mathbf{x}]$). But any $P(gP(h)) - P(hP(g))$ can be generated by $P(x_k P(x_i x_j)) - P(x_i P(x_k x_j))$'s. $\square$

We note that $C(a; j, (i,k)) + C(a; j, (k,i)) = 0$ so from now on we identify $C(a; j, (i,k))$ with $-C(a; j, (k,i))$.

Lemma 5. *In fact, $I_{\mathcal{P}}$ is generated by $C(a; j, (i,k))$'s ($1 \le a \le d, \ \ 1 \le i,j,k \le d$).*

Proof. It is enough to prove that for any $1 \le i, j, k \le d$, the polynomial $C(0; j, (i,k))$ is generated by $C(a; b, (e,f))$'s $(1 \le a, b, e, f \le d)$. Fix any u, $1 \le u \le d$. Then we have

$$
\begin{aligned}
C(0; j,(i,k)) &= \sum_{m=1}^{d} (p_{m,ij}p_{0,km} - p_{m,kj}p_{0,im}) \\
= -\sum_{m=1}^{d} \Big(p_{m,ij} &\sum_{t=1}^{d}(p_{t,km}p_{u,tu} - p_{t,ku}p_{u,tm}) - p_{m,kj}\sum_{t=1}^{d}(p_{t,im}p_{u,tu} - p_{t,iu}p_{u,tm})\Big) \\
&+ \sum_{m=1}^{d}\Big(p_{m,ij}C(u; k, (m,u)) - p_{m,kj}C(u; i, (m,u))\Big)
\end{aligned}
$$

$$
\begin{aligned}
= -\sum_{t=1}^{d}\Big(p_{u,tu}&\sum_{m=1}^{d}(p_{m,ij}p_{t,km} - p_{m,kj}p_{t,im}) \\
&-p_{t,ku}\sum_{m=1}^{d}(p_{m,ij}p_{u,tm} - p_{m,it}p_{u,jm}) + p_{t,iu}\sum_{m=1}^{d}(p_{m,kj}p_{u,tm} - p_{m,kt}p_{u,jm})\Big) \\
&+ \sum_{m=1}^{d} p_{u,jm}\sum_{t=1}^{d}(p_{t,ku}p_{m,it} - p_{t,iu}p_{m,kt}) \\
&+ \sum_{m=1}^{d}\Big(p_{m,ij}C(u; k, (m,u)) - p_{m,kj}C(u; i, (m,u))\Big) \\
= -\sum_{t=1}^{d}\Big(p_{u,tu}&C(t; j, (i,k)) - p_{t,ku}C(u; i, (j,t)) + p_{t,iu}C(u; k, (j,t))\Big) \\
&+ \sum_{m=1}^{d} p_{u,jm}C(m; u, (k,i)) \\
&+ \sum_{m=1}^{d}\Big(p_{m,ij}C(u; k, (m,u)) - p_{m,kj}C(u; i, (m,u))\Big).
\end{aligned}
$$

□

So the set of generators of $I_{\mathcal{P}}$ is

$$\{C(a; j, (i,k)) \mid 1 \le a, i, j, k \le d)\}.$$

We associate to this a representation of $GL(V)$.

Proposition 6. *The $\mathbb{C}$-vector space W of generators*

$$\frac{< C(a; j, (i,k)) \mid 1 \le a, i, j, k \le d) >}{C(a; j, (i,k)) + C(a; j, (k,i))}$$

is canonically isomorphic to

$$\mathbb{S}_{(3,2,1,\cdots,1,0)}V \bigoplus \mathbb{S}_{(3,1,1,\cdots,1,1)}V$$

as $\mathbb{C}$*-vector spaces, where* V *is a* d*-dimensional vector space and* $\mathbb{S}_{(3,2,1,\cdots,1,0)}$ (*resp.* $\mathbb{S}_{(3,1,1,\cdots,1,1)}$) *is the Schur functor corresponding to the partition* $(3,2,1,\cdots,1,0)$ (*resp.* $(3,1,1,\cdots,1,1)$) *of* $(d+2)$.

Proof. Let $V = \bigoplus_{i=1}^d \mathbb{C}v_i$. Define

$$\varphi : W \longrightarrow \bigwedge^{d-1} V \otimes V \otimes \bigwedge^2 V$$

by

$$\varphi : C(a; j, (i,k)) \mapsto (-1)^a (v_1 \wedge \cdots \wedge \hat{v_a} \wedge \cdots \wedge v_d) \otimes v_j \otimes (v_i \wedge v_k).$$

Then it is clear that φ is injective.

By Littlewood-Richardson rule, we have

$$\begin{aligned}
&\bigwedge^{d-1} V \otimes V \otimes \bigwedge^2 V \\
&\cong \mathbb{S}_{(1,1,1,\cdots,1,0)}V \otimes V \otimes \mathbb{S}_{(1,1,0,\cdots,0,0)}V \\
&\cong \mathbb{S}_{(3,2,1,\cdots,1,0)}V \bigoplus \mathbb{S}_{(3,1,1,\cdots,1,1)}V \bigoplus (\mathbb{S}_{(2,2,1,\cdots,1,1)}V)^{\oplus 2} \bigoplus \mathbb{S}_{(2,2,2,1,\cdots,1,0)}V \\
&\cong \mathbb{S}_{(3,2,1,\cdots,1,0)}V \bigoplus \mathbb{S}_{(3,1,1,\cdots,1,1)}V \bigoplus \bigwedge^d V \otimes \bigwedge^2 V \bigoplus \bigwedge^{d-1} V \otimes \bigwedge^3 V,
\end{aligned}$$

where each partition is of $(d+2)$. We will show that the images of W under φ lie neither on $\bigwedge^d V \otimes \bigwedge^2 V$ nor $\bigwedge^{d-1} V \otimes \bigwedge^3 V$.

Since

$$\sum_{j=1}^d (-1)^j (v_1 \wedge \cdots \wedge \hat{v_j} \wedge \cdots \wedge v_d) \otimes v_j \otimes (v_i \wedge v_k), \quad 1 \le i < k \le d,$$

generate $\bigwedge^d V \otimes \bigwedge^2 V$, we need to show that

$$\sum_{j=1}^d C(j; j, (i,k)) = 0. \tag{3}$$

But this is elementary because

$$\sum_{j=1}^d C(j; j, (i,k)) = \sum_{j=1}^d \sum_{m=1}^d (p_{m,ij}p_{j,km} - p_{m,kj}p_{j,im}) = 0.$$

Since

$$\begin{aligned}&(v_1\wedge\cdots\wedge\hat{v_a}\wedge\cdots\wedge v_d)\otimes v_j\otimes(v_i\wedge v_k)\\&+(v_1\wedge\cdots\wedge\hat{v_a}\wedge\cdots\wedge v_d)\otimes v_k\otimes(v_j\wedge v_i)\\&+(v_1\wedge\cdots\wedge\hat{v_a}\wedge\cdots\wedge v_d)\otimes v_i\otimes(v_k\wedge v_j),\quad 1\le a\le d,\ 1\le j<i<k\le d,\end{aligned}$$

generate $\bigwedge^{d-1}V\otimes\bigwedge^3 V$, we need to show that

$$C(a;j,(i,k))+C(a;k,(j,i))+C(a;i,(k,j))=0. \tag{4}$$

But this is again elementary because

$$\begin{aligned}&\sum_{m=1}^{d}(p_{m,ij}p_{a,km}-p_{m,kj}p_{a,im})\\&+\sum_{m=1}^{d}(p_{m,jk}p_{a,im}-p_{m,ik}p_{a,jm})\\&+\sum_{m=1}^{d}(p_{m,ki}p_{a,jm}-p_{m,ji}p_{a,km})=0.\end{aligned}$$

Therefore $\varphi(W)\subset\mathbb{S}_{(3,2,1,\cdots,1,0)}V\bigoplus\mathbb{S}_{(3,1,1,\cdots,1,1)}V$, in other words,

$$\varphi: W\longrightarrow\mathbb{S}_{(3,2,1,\cdots,1,0)}V\bigoplus\mathbb{S}_{(3,1,1,\cdots,1,1)}V$$

is injective.

The next lemma completes the proof. □

Lemma 7. *$\varphi: W\longrightarrow\mathbb{S}_{(3,2,1,\cdots,1,0)}V\bigoplus\mathbb{S}_{(3,1,1,\cdots,1,1)}V$ is surjective.*

Proof. It is enough to show that there are no other nontrivial $\mathbb{C}$-linear relations among $C(a;j,(i,k))$'s than $\mathbb{C}$-linear combinations of (3) and (4).

Suppose

$$C(a;j,(i,k))+\sum_{u,b,e,f}c_{u,b,(e,f)}C(u;b,(e,f))=0,\qquad c_{u,b,(e,f)}\in\mathbb{C}. \tag{5}$$

If $a\neq i,j,k$ then $C(a;j,(i,k))$ contains a term $p_{m,ij}p_{a,km}$ and a term $p_{m,kj}p_{a,im}$. The term $p_{m,ij}p_{a,km}$ appears only in $C(a;j,(i,k))$ and $C(a;i,(k,j))$ among all $C(u;b,(e,f))$, $1\le u,b,e,f\le d$. Similarly the term $p_{m,kj}p_{a,im}$ appears only in $C(a;j,(i,k))$ and $C(a;k,(j,i))$. So the left hand side of (5) must be a nontrivial linear combination of (4) and other relations.

Similarly even if $a=i,j,$ or k, each term in $C(a;j,(i,k))$ appears only in the ones involved in (3) or (4). To get cancelation among these, the left hand side of (5) must contain (3) or (4). Repeating the argument, (5) becomes a linear combination of (3) and (4). □

Lemma 8. *We have*

$$\bigwedge^{d-1} V \otimes \mathrm{Sym}^2 V \cong \mathbb{S}_{(2,1,1,\cdots,1,1)}V \oplus \mathbb{S}_{(3,1,1,\cdots,1,0)}V,$$

and

$$\begin{aligned}\mathrm{Sym}^2(\mathbb{S}_{(3,1,1,\cdots,1,0)}V) \cong & \mathbb{S}_{(6,2,2,\cdots,2,0)}V \oplus \mathbb{S}_{(5,3,2,\cdots,1,1)}V \oplus \mathbb{S}_{(5,2,2,\cdots,2,1)}V \\ & \oplus \mathbb{S}_{(4,4,2,\cdots,2,0)}V \oplus \mathbb{S}_{(4,3,2,\cdots,2,1)}V \oplus \mathbb{S}_{(4,2,2,\cdots,2,2)}V.\end{aligned}$$

(If $d = 3$ then $\mathbb{S}_{(5,3,2,\cdots,1,1)}V$ does not appear.)

Proof. The first isomorphism follows from the Littlewood-Richardson rule. The second isomorphism can be calculated by [2, pp.124–128]. □

Lemma 9. *There is an injective homomorphism*

$$j : \mathbb{S}_{(4,3,2,\cdots,2,1)}V \hookrightarrow \mathrm{Sym}^2(\bigwedge^{d-1} V \otimes \mathrm{Sym}^2 V)$$

such that $\mathcal{P}$ (hence the symmetric affine subscheme U) is isomorphic to

$$\mathrm{Spec}\frac{\mathrm{Sym}^{\bullet}(\bigwedge^{d-1} V \otimes \mathrm{Sym}^2 V)}{\mathbb{S}_{(4,3,2,\cdots,2,1)}V},$$

where $(4, 3, 2, \cdots, 2, 1)$ is a partition of $(2d + 2)$.

Proof. Consider a diagram

$$\begin{array}{ccc} \frac{\mathbb{C}[p'_{0,ij},\, p'_{k,st}]_{1\le i,j,k,s,t\le d}}{(p'_{0,ij}-p'_{0,ji},\, p'_{k,st}-p'_{k,ts})} & \xleftarrow{\ f\ } & \frac{\mathbb{C}[p_{0,ij},\, p_{k,st}]_{1\le i,j,k,s,t\le d}}{(p_{0,ij}-p_{0,ji},\, p_{k,st}-p_{k,ts})} =: R \\ \big\downarrow g & & \\ T := \frac{\mathbb{C}[p'_{k,st}]_{1\le k,s,t\le d}}{(p'_{k,st}-p'_{k,ts})} & & \end{array}$$

where g is the natural projection and f^{-1} is defined by

$$\begin{aligned} p'_{0,ij} &\mapsto C(i+1; j, (i, i+1)), && 1 \le i \le j \le d, \\ & \qquad (\text{if } i = d \text{ then } i + 1 := 1) && \\ p'_{k,st} &\mapsto p_{k,st}, && 1 \le s \le t \le d,\ 1 \le k \le d. \end{aligned}$$

In fact f is an isomorphism because $p_{0,ij}$ is a linear term in

$$C(i+1; j, (i, i+1)) = p_{0,ij} + \sum_{m=1}^{d} (p_{m,ij}p_{(i+1),(i+1)m} - p_{m,(i+1)j}p_{(i+1),im}).$$

Since $C(i+1; j, (i, i+1)) \in I_{\mathcal{P}}$, we have an induced isomorphism

$$\frac{R}{I_{\mathcal{P}}} \cong \frac{T}{I_{\mathcal{P}}T}, \tag{6}$$

where $I_{\mathcal{P}}T$ is the expansion of $I_{\mathcal{P}}$ to T. We note that in this construction $C(i+1; j, (i, i+1))$ can be replaced by any $C(k; j, (i, k))$ or $C(k; i, (j, k))$ $(k \neq i, j)$, because the resulting $I_{\mathcal{P}}T$ does not depend on the choice $C(k; j, (i, k))$ or $C(k; i, (j, k))$. In fact this construction is natural in the sense that we eliminate all the linear terms appearing in $C(a; j, (i, k))$ so that the ideal $I_{\mathcal{P}}T$ is generated by quadratic equations.

Since $p'_{0,ij}$ are eliminated under passing g, the direct summand $\mathbb{S}_{(3,1,1,\cdots,1,1)}V (\cong \mathrm{Sym}^2 V)$ in W is eliminated. Then, by Proposition 6, the vector space of generators of $I_{\mathcal{P}}T$ is canonically isomorphic to $\mathbb{S}_{(3,2,1,\cdots,1,0)}V$ hence to

$$\bigwedge^d V \otimes \mathbb{S}_{(3,2,1,\cdots,1,0)}V \cong \mathbb{S}_{(4,3,2,\cdots,2,1)}V \subset \mathrm{Sym}^2(\bigwedge^{d-1} V \otimes \mathrm{Sym}^2 V),$$

where the last containment follows from Lemma 8.

The isomorphism of rings

$$T = \frac{\mathbb{C}[p'_{k,st}]_{1 \le k,s,t \le d}}{(p'_{k,st} - p'_{k,ts})} \cong \mathrm{Sym}^\bullet(\bigwedge^{d-1} V \otimes \mathrm{Sym}^2 V)$$

naturally induces the isomorphism of quotient rings

$$\frac{T}{I_{\mathcal{P}}T} \cong \frac{\mathrm{Sym}^\bullet(\bigwedge^{d-1} V \otimes \mathrm{Sym}^2 V)}{\mathbb{S}_{(4,3,2,\cdots,2,1)}V}. \tag{7}$$

Combining this with (6) gives the desired result. □

Theorem 10. *There is an injective homomorphism*

$$j : \mathbb{S}_{(4,3,2,\cdots,2,1)}V \hookrightarrow \mathrm{Sym}^2(\mathbb{S}_{(3,1,1,\cdots,1,0)}V)$$

such that $\mathcal{P}$ (hence the symmetric affine subscheme U) is isomorphic to

$$\mathbb{C}^d \times \mathrm{Spec} \frac{\mathrm{Sym}^\bullet(\mathbb{S}_{(3,1,1,\cdots,1,0)}V)}{\mathbb{S}_{(4,3,2,\cdots,2,1)}V},$$

where $(3, 1, 1, \cdots, 1, 0)$ is a partition of $(d+1)$ and $(4, 3, 2, \cdots, 2, 1)$ is of $(2d+2)$.

Sketch of Proof. Define an isomorphism of rings

$$T = \frac{\mathbb{C}[p'_{k,st}]_{1 \le k,s,t \le d}}{(p'_{k,st} - p'_{k,ts})} \xrightarrow{\cong} \frac{\mathbb{C}[q_{k,st}]_{1 \le k,s,t \le d}}{(q_{k,st} - q_{k,ts})} =: Q$$

by

$$p'_{k,st} \mapsto \begin{cases} q_{k,sk} + q_{s,ss}, & \text{if } k = t \\ q_{k,st}, & \text{if } k \neq s, t. \end{cases}$$

As a matter of fact this is a natural isomorphism, because the square of any maximal ideal in $\mathbb{C}[\mathbf{x}]$ satisfies $p'_{k,sk} - \frac{1}{2}p'_{s,ss} = 0$ $(k \neq s)$, i.e. $q_{k,sk} = 0$. It is straightforward to check that no element in minimal generators of $I_{\mathcal{P}}Q$ contains terms involving $q_{s,ss}$, $1 \leq s \leq d$. For example, if a, i, j, k are distinct, then

$$\begin{aligned} C(a; j, (i,k)) &= \sum_{m=1}^{d} (p_{m,ij}p_{a,km} - p_{m,kj}p_{a,im}) \\ &= \sum_{m \neq a,j,i,k} (p_{m,ij}p_{a,km} - p_{m,kj}p_{a,im}) \\ &\quad + (p_{j,ij}p_{a,kj} - p_{j,kj}p_{a,ij}) + (p_{a,ij}p_{a,ka} - p_{a,kj}p_{a,ia}) \\ &\quad + (p_{i,ij}p_{a,ki} - p_{i,kj}p_{a,ii}) + (p_{k,ij}p_{a,kk} - p_{k,kj}p_{a,ik}) \end{aligned}$$

becomes

$$\begin{aligned} &\sum_{m \neq a,j,i,k} (q_{m,ij}q_{a,km} - q_{m,kj}q_{a,im}) \\ &\quad + ((q_{j,ij} + q_{i,ii})q_{a,kj} - (q_{j,kj} + q_{k,kk})q_{a,ij}) + (q_{a,ij}(q_{a,ka} + q_{k,kk}) - q_{a,kj}(q_{a,ia} + q_{i,ii})) \\ &\quad + ((q_{i,ij} + q_{j,jj})q_{a,ki} - q_{i,kj}q_{a,ii}) + (q_{k,ij}q_{a,kk} - (q_{k,kj} + q_{j,jj})q_{a,ik}) \\ &= \sum_{m \neq a,j,i,k} (q_{m,ij}q_{a,km} - q_{m,kj}q_{a,im}) \\ &\quad + (q_{j,ij}q_{a,kj} - q_{j,kj}q_{a,ij}) + (q_{a,ij}q_{a,ka} - q_{a,kj}q_{a,ia}) \\ &\quad + (q_{i,ij}q_{a,ki} - q_{i,kj}q_{a,ii}) + (q_{k,ij}q_{a,kk} - q_{k,kj}q_{a,ik}), \end{aligned}$$

in which no term involves $q_{s,ss}$, $1 \leq s \leq d$.

Therefore we get

$$\frac{T}{I_{\mathcal{P}}T} \cong \frac{Q}{I_{\mathcal{P}}Q} \cong \mathbb{C}[q_{s,ss}]_{1 \leq s \leq d} \otimes_{\mathbb{C}} \frac{\mathbb{C}[q_{k,st}]_{1 \leq k,s,t \leq d,\ \ k \neq s \text{ or } t \neq s}}{(q_{k,st} - q_{k,ts})} \Big/ I_{\mathcal{P}}Q.$$

On the other hand, Lemma 8 implies

$$\mathrm{Sym}^{\bullet}(\bigwedge^{d-1} V \otimes \mathrm{Sym}^2 V) \cong \mathrm{Sym}^{\bullet}(\mathbb{S}_{(2,1,1,\cdots,1,1)}V \oplus \mathbb{S}_{(3,1,1,\cdots,1,0)}V).$$

We may identify the basis of $\mathbb{S}_{(2,1,1,\cdots,1,1)}V$ with $\{q_{s,ss} | 1 \leq s \leq d\}$. So, by (7), we have

$$\begin{aligned} \frac{T}{I_{\mathcal{P}}T} &\cong \mathbb{C}[q_{s,ss}]_{1 \leq s \leq d} \otimes_{\mathbb{C}} \frac{\mathbb{C}[q_{k,st}]_{1 \leq k,s,t \leq d,\ \ k \neq s \text{ or } t \neq s}}{(q_{k,st} - q_{k,ts})} \Big/ I_{\mathcal{P}}Q \\ &\cong \mathrm{Sym}^{\bullet}(\mathbb{S}_{(2,1,1,\cdots,1,1)}V) \otimes \frac{\mathrm{Sym}^{\bullet}(\mathbb{S}_{(3,1,1,\cdots,1,0)}V)}{\mathbb{S}_{(4,3,2,\cdots,2,1)}V}. \end{aligned}$$

Combining this with (6) gives the desired result. $\square$

3 Proof of Theorem A

The following lemma is elementary and well-known (for example, see [11, Theorem 18.32]). For the convenience of the reader, we include its proof here.

Lemma 11. *If* $\mathrm{Hilb}^{d+1}(\mathbb{C}^d)$ *is reducible then so is* $\mathrm{Hilb}^n(\mathbb{C}^d)$ *for* $n \geq d+1$.

Proof. Let $H_n^d := \mathrm{Hilb}^n(\mathbb{C}^d)$ and $R_n^d \subset H_n^d$ denote the closure of the open set parametrizing radical ideals. It is enough to show that $R_n^d \neq H_n^d$ implies $R_{n+1}^d \neq H_{n+1}^d$. If $I \in H_n^d \setminus R_n^d$ and $P = (p_1, ..., p_d) \in \mathbb{C}^d$ is not a zero of I, then $I \cap \langle x_1 - p_1, ..., x_d - p_d \rangle$ is a point in $H_{n+1}^d \setminus R_{n+1}^d$. $\square$

Proof of Theorem A. Due to Lemma 11, it suffices to show that $\mathrm{Hilb}^{d+1}(\mathbb{C}^d)$ is reducible for $d \geq 12$. Actually we will prove that the most symmetric open affine subscheme U of $\mathrm{Hilb}^{d+1}(\mathbb{C}^d)$ is reducible for $d \geq 12$.

The symmetric open affine subscheme U of $\mathrm{Hilb}^{14}(\mathbb{C}^{13})$ is reducible. In fact it can be obtained by modifying Iarrobino's and Shekhtman's constructions ([9], [13]). To each 36×5 matrix B over $\mathbb{C}$, we associate an ideal

$$I_B := \begin{pmatrix} x_6^2 + b_{1,1}x_1 + b_{1,2}x_2 + \cdots + b_{1,5}x_5, \\ x_6x_7 + b_{2,1}x_1 + b_{2,2}x_2 + \cdots + b_{2,5}x_5, \\ \vdots \\ x_{13}^2 + b_{36,1}x_1 + b_{36,2}x_2 + \cdots + b_{36,5}x_5 \end{pmatrix} + (x_1, \cdots, x_5) \cap \mathfrak{m}^2 + \mathfrak{m}^3,$$

where $\mathfrak{m}$ denotes the maximal ideal corresponding to the origin. Note that $\{1, x_1, ..., x_{13}\}$ is a $\mathbb{C}$-basis of $\mathbb{C}[x_1, ..., x_{13}]/I_B$. Then the dimension of the family of I_B's is $36 \cdot 5 = 180$. Applying affine translations, we get a $193(= 180 + 13)$-dimensional family of ideals. Since $193 > 13 \cdot 14$(=the dimension of the closure of the set of radical ideals), U is reducible.

This construction can be easily generalized to $d \geq 13$. So the symmetric open affine subscheme of $\mathrm{Hilb}^{d+1}(\mathbb{C}^d)$ is reducible for $d \geq 13$. For the case of $d = 12$, we use 36×4 matrices and get a $156(= 36 \cdot 4 + 12)$-dimensional family $\mathcal{F}$ of ideals. Then $\mathcal{F}$ has the same dimension $(156 = 12 \cdot 13)$ as the closure of the set of radical ideals, but a general member in $\mathcal{F}$ is not a radical ideal. So $\mathcal{F}$ is not contained in the principal (radical) component. $\square$

It would be interesting to find equations for the principal (radical) component - the component containing radical ideals - of its symmetric open affine subscheme. The ideal defining the principal component contains the ideal generated by $\mathbb{S}_{(4,3,2,\cdots,2,1)}V$.

4 Questions and Examples

It is well known ([10]) that if $d = 3$ then U is isomorphic to a cone over the Plücker embedding of the Grassmannian $G(2, 6)$ with a three-dimensional vertex. So we have the following :

Remark 12. Let V be a 3-dimensional vector space and W a 6-dimensional vector space. Then

$$\frac{\mathrm{Sym}^\bullet(\mathbb{S}_{(3,1,0)}V)}{\mathbb{S}_{(4,3,1)}V} \cong \frac{\mathrm{Sym}^\bullet(\bigwedge^2 W)}{\bigwedge^4 W}.$$

□

Generalizing Theorem B, we have

Conjecture D. *Let $d \geq 3$ and $n = \binom{d+m}{m}$ for some positive integer m. Let $U_m \subset \mathrm{Hilb}^n(\mathbb{C}^d)$ denote the affine open subscheme consisting of all ideals $I \in \mathrm{Hilb}^n(\mathbb{C}^d)$ such that $\{\mathbf{x}^{\mathbf{u}} \mid |\mathbf{u}| \leq m\} = \{1, x_1, ..., x_d^m\}$ is a $\mathbb{C}$-basis of $\mathbb{C}[\mathbf{x}]/I$. Then there are injective homomorphisms*

$$j_k : \mathbb{S}_{(3k+1,2k+1,2k,\cdots,2k,k)}V \hookrightarrow \mathrm{Sym}^2(\mathbb{S}_{(2k+1,k,k,\cdots,k,0)}V), \qquad 1 \leq k \leq m,$$

of Schur modules such that U_m is isomorphic to the induced scheme by j_k

$$\mathbb{C}^d \times \prod_{k=1}^{m} \mathrm{Spec}\frac{\mathrm{Sym}^\bullet(\mathbb{S}_{(2k+1,k,k,\cdots,k,0)}V)}{\mathbb{S}_{(3k+1,2k+1,2k,\cdots,2k,k)}V},$$

where V is a d-dimensional vector space over $\mathbb{C}$, $(2k+1, k, k, \cdots, k, 0)$ is a partition of $k(d+1)$ and $(3k+1, 2k+1, 2k, \cdots, 2k, k)$ is of $2k(d+1)$.

One of possible ways to obtain the dimension of U might be to find its Hilbert polynomial.

Lemma 13. *Let $H(r)$ be the Hilbert function of $\frac{\mathrm{Sym}^\bullet(\mathbb{S}_{(3,1,1,\cdots,1,0)}V)}{\mathbb{S}_{(4,3,2,\cdots,2,1)}V}$. Then*

$$H(1) = d\binom{d+1}{2} - d$$

and

$$H(2) = \binom{d\binom{d+1}{2} - d + 1}{2} - \frac{d^2(d^2-4)}{3}.$$

Proof. We recall the fact that if $\mu = (\mu_1 \geq \cdots \geq \mu_d \geq 0)$ then

$$\dim \mathbb{S}_\mu = \prod_{1 \leq i < j \leq d} \frac{\mu_i - \mu_j + j - i}{j - i}$$

(for example, see [5, Theorem 6.3]). It is straightforward to check that

$$\dim \mathbb{S}_{(3,1,1,\cdots,1,0)}V = d\binom{d+1}{2} - d$$

and

$$\dim \mathbb{S}_{(4,3,2,\cdots,2,1)}V = \frac{d^2(d^2-4)}{3}.$$

□

Conjecture 14. *Let $\tilde{H}(r)$ be the Hilbert polynomial of $\frac{\mathrm{Sym}^\bullet(\mathbb{S}_{(3,1,1,\cdots,1,0)}V)}{\mathbb{S}_{(4,3,2,\cdots,2,1)}V}$. Then $\tilde{H}(r) = H(r)$ for every $r \geq 0$.*

Remark 15. The case of $d = 3$ is well-known(cf. [10]). Actual equations $C(a; j, (i, k))$ (or [10, p242]) are relatively simple so we can use the computer algebra system Macaulay 2. It shows that if $d = 3$ then

$$\tilde{H}(r) = H(r) = 14\binom{r+8}{8} - 21\binom{r+7}{7} + 9\binom{r+6}{6} - \binom{r+5}{5}.$$

Conjecture 16. *If $d \geq 4$ then*

$$\begin{aligned} H(3) &= \dim\ \mathrm{Sym}^3(\mathbb{S}_{(3,1,1,\cdots,1,0)}V) - \dim\ \mathbb{S}_{(4,3,2,\cdots,2,1)}V \otimes \mathbb{S}_{(3,1,1,\cdots,1,0)}V \\ &\quad + \dim\ \left(\mathbb{S}_{(6,4,3,\cdots,3,2)}V \oplus \mathbb{S}_{(5,4,4,3,\cdots,3,2)}V \oplus \mathbb{S}_{(5,4,3,\cdots,3,3)}V\right) \\ &= \binom{d\binom{d+1}{2} - d + 2}{3} - \frac{d^2(d^2-4)\left(d\binom{d+1}{2} - d\right)}{3} + \frac{d(d^2-4)(3d^2+1)}{12}. \end{aligned}$$

Remark 17. The conjecture holds true for $d = 3$. When $d = 4$ or 5, it coincides with the results obtained by using actual equations $C(a; j(i, k))$ and running Macaulay 2. The three Schur modules $\mathbb{S}_{(6,4,3,\cdots,3,2)}V$, $\mathbb{S}_{(5,4,4,3,\cdots,3,2)}V$, and $\mathbb{S}_{(5,4,3,\cdots,3,3)}V$ appear with multiplicity > 1 in the decomposition of $\mathbb{S}_{(4,3,2,\cdots,2,1)}V \otimes \mathbb{S}_{(3,1,1,\cdots,1,0)}V$.

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